

Roy–Steiner equations for πN scattering

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Starting from hyperbolic dispersion relations for the invariant amplitudes of pion–nucleon scattering together with crossing symmetry and unitarity, one can derive a closed system of integral equations for the partial waves of both the s -channel ($\pi N \rightarrow \pi N$) and the t -channel ($\pi\pi \rightarrow \bar{N}N$) reaction, called Roy–Steiner equations. After giving a brief overview of the Roy–Steiner system for πN scattering, we demonstrate that the solution of the t -channel subsystem, which represents the first step in solving the full system, can be achieved by means of Muskhelishvili–Omnès techniques. In particular, we present results for the P -waves featuring in the dispersive analysis of the electromagnetic form factors of the nucleon.

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1. Introducing Roy–Steiner equations for πN scattering

Partial-wave dispersion relations (PWDRs) together with unitarity and crossing symmetry as well as isospin and chiral symmetry (i.e. all available symmetry constraints) have repeatedly proven to be a powerful tool for studying processes at low energies with high precision [1–4]. For πN scattering the (unsubtracted) hyperbolic dispersion relations (HDRs) for the usual Lorentz-invariant amplitudes read [5] (using the notation of [6], see [7] for more details)

$$\begin{aligned} A^+(s, t) &= \frac{1}{\pi} \int_{s_+}^{\infty} ds' \left[\frac{1}{s' - s} + \frac{1}{s' - u} - \frac{1}{s' - a} \right] \text{Im} A^+(s', t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\text{Im} A^+(s', t')}{t' - t}, \\ B^+(s, t) &= N^+(s, t) + \frac{1}{\pi} \int_{s_+}^{\infty} ds' \left[\frac{1}{s' - s} - \frac{1}{s' - u} \right] \text{Im} B^+(s', t') + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \frac{s - u}{s' - u'} \frac{\text{Im} B^+(s', t')}{t' - t}, \\ N^+(s, t) &= g^2 \left[\frac{1}{m^2 - s} - \frac{1}{m^2 - u} \right], \quad (s - a)(u - a) = b = (s' - a)(u' - a), \end{aligned} \quad (1.1)$$

and similarly for A^- , B^- , and N^- , where N^\pm are the nucleon pole terms and the “external” (unprimed) and “internal” (primed) kinematics are related by real hyperbola parameters a and b (as well as via $s + t + u = 2(m^2 + M_\pi^2) = s' + t' + u'$), so that HDRs allow for the combination of all physical regions, which is known to be crucial for a reliable continuation into the subthreshold region and hence for an accurate determination of the πN σ -term. Furthermore, the imaginary parts are only needed in regions where the corresponding partial-wave decompositions converge and the range of convergence can be maximized by tuning the free hyperbola parameter a . While the s -channel integrals start at the threshold $s_+ = W_+^2 = (m + M_\pi)^2$, the t -channel contributes already above the pseudothreshold $t_\pi = 4M_\pi^2$ far below the threshold $t_N = 4m^2$. Depending on the asymptotic behavior of the imaginary parts, in principle it could be necessary to subtract the HDRs to ensure the convergence of the integrals, thereby parameterizing high-energy information with polynomials containing a priori unknown subtraction constants. However, (additional) subtractions may also be introduced to lessen the dependence of the low-energy solution on high-energy input; the corresponding subtraction parameters then obey respective sum rules. For πN scattering it proves particularly useful to subtract at the subthreshold point ($s = u, t = 0$), as this preserves the $s \leftrightarrow u$ crossing symmetry (which can be made explicit in terms of the crossing variable $v = (s - u)/(4m)$ via $D^\pm(v, t) = A^\pm + vB^\pm = \pm D^\pm(-v, t)$). This is especially favorable for the t -channel subproblem and facilitates matching to chiral perturbation theory [8, 9] to determine the subtraction constants, which thus can be identified with the subthreshold expansion parameters.¹ In addition to the presentation in [7], we also introduce a (partial) third subtraction, which is related to the parameters a_{10}^+ and a_{01}^- of the subthreshold expansions (with $d_{0n}^+ = a_{0n}^+$ for all $n \geq 0$)

$$A^+(v, t) = \frac{g^2}{m} + d_{00}^+ + d_{01}^+ t + a_{10}^+ v^2 + \mathcal{O}(v^4, v^2 t, t^2), \quad A^-(v, t) = a_{00}^- v + a_{01}^- v t + \mathcal{O}(v^3, v t^2). \quad (1.2)$$

¹For the PWDRs of $\pi\pi$ scattering, called Roy equations [10], an analogous matching procedure for the $\pi\pi$ scattering lengths as pertinent subtraction parameters has been conducted in [11]. In contrast to $\pi\pi$ scattering, the πN scattering lengths can be extracted with high accuracy from hadronic-atom data [12, 13] and may thus serve as additional constraints on the subtraction constants in the Roy–Steiner system.

In order to derive the partial-wave HDRs, called Roy–Steiner (RS) equations, one needs to expand the s - and t -channel imaginary parts in (1.1) into the respective partial waves and subsequently project the full expanded equations onto either s - or t -channel partial waves; the resulting sets of integral equations together with the respective partial-wave unitarity relations then form the s - and t -channel RS subsystems. According to [5], the (unsubtracted) s -channel RS equations read (based on the MacDowell symmetry $f_{(\ell+1)-}^J(W) = -f_{\ell+}^J(-W)$ for all $\ell \geq 0$ [14])

$$f_{\ell+}^J(W) = N_{\ell+}^J(W) + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_J \left\{ G_{\ell J}(W, t') \operatorname{Im} f_{\ell+}^J(t') + H_{\ell J}(W, t') \operatorname{Im} f_{\ell-}^J(t') \right\} \\ + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{\ell'=0}^{\infty} \left\{ K_{\ell \ell'}^J(W, W') \operatorname{Im} f_{\ell'+}^J(W') + K_{\ell \ell'}^J(W, -W') \operatorname{Im} f_{(\ell'+1)-}^J(W') \right\}, \quad (1.3)$$

where due to G -parity only even/odd J contribute for isospin $I = +/ -$, respectively, and the partial-wave projections of the pole terms as well as the (lowest) kernels are analytically known, the latter including in particular the Cauchy kernel: $K_{\ell \ell'}^J(W, W') = \delta_{\ell \ell'} / (W' - W) + \dots$. The s -channel $I = \pm$ partial waves are intertwined by the usual unitarity relations, which are diagonal in the s -channel isospin basis $I_s \in \{1/2, 3/2\}$ only. Once the t -channel partial waves are known, the structure of the s -channel RS subsystem is therefore similar to the $\pi\pi$ Roy equations, cf. [1]. As shown in [7], the corresponding (unsubtracted) t -channel RS equations are given by

$$f_{\ell+}^J(t) = \tilde{N}_{\ell+}^J(t) + \frac{1}{\pi} \int_{W_+}^{\infty} dW' \sum_{\ell'=0}^{\infty} \left\{ \tilde{G}_{J\ell}(t, W') \operatorname{Im} f_{\ell'+}^J(W') + \tilde{G}_{J\ell}(t, -W') \operatorname{Im} f_{(\ell'+1)-}^J(W') \right\} \\ + \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \sum_{J'} \left\{ \tilde{K}_{JJ'}^1(t, t') \operatorname{Im} f_{\ell'+}^{J'}(t') + \tilde{K}_{JJ'}^2(t, t') \operatorname{Im} f_{\ell-}^{J'}(t') \right\} \quad (1.4)$$

and similarly for the $f_{\ell-}^J$ except for the fact that these do not receive contributions from the $f_{\ell+}^J$. Here, only even or odd J' couple to even or odd J (corresponding to t -channel isospin $I_t = 0$ or $I_t = 1$), respectively, and $\tilde{K}_{JJ'}^1$ (as well as the analogous $\tilde{K}_{JJ'}^3$ for the $f_{\ell-}^J$) contains the Cauchy kernel. Moreover, it turns out that only higher t -channel partial waves contribute to lower ones. Assuming Mandelstam analyticity, the equations (1.4) are valid for $\sqrt{t} \in [2M_\pi, 2.00 \text{ GeV}]$ using $a = -2.71 M_\pi^2$, whereas (1.3) holds for $W \in [m + M_\pi, 1.38 \text{ GeV}]$ using $a = -23.19 M_\pi^2$. The t -channel unitarity relations are diagonal in I_t and only linear in the f_{\pm}^J (below the first inelastic threshold t_{inel})

$$\operatorname{Im} f_{\pm}^J(t) = \sigma_t^\pi (t_J^I(t))^* f_{\pm}^J(t) \theta(t - t_\pi), \quad \sigma_t^\pi t_J^I(t) = \sin \delta_J^I(t) e^{i\delta_J^I(t)}, \quad \sigma_t^\pi(t) = \sqrt{1 - t_\pi/t},$$

from which one can infer Watson’s final state interaction theorem [15] stating that (in the “elastic” region) the phase of f_{\pm}^J is given by the phase δ_J^I of the respective $\pi\pi$ scattering partial wave t_J^I .

Due to the simpler recoupling scheme for the f_{\pm}^J , the t -channel RS subsystem can be recast as a (single-channel) Muskhelishvili–Omnès (MO) problem [16] with a finite matching point t_m [3] for f_{\pm}^0 , f_{\pm}^J , and the linear combinations $\Gamma^J(t) = m\sqrt{J/(J+1)} f_{\pm}^J(t) - f_{\pm}^J(t)$ with $\Gamma^J(t_N) = 0$ for all $J \geq 1$ of the generic form (the details are given in [7])

$$f(t) = \Delta(t) + \frac{1}{\pi} \int_{t_\pi}^{t_m} dt' \frac{\sin \delta(t') e^{-i\delta(t')} f(t')}{t' - t} + \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{\operatorname{Im} f(t')}{t' - t} \equiv |f(t)| e^{i\delta(t)} \quad \text{for } t \leq t_m < t_{\text{inel}},$$

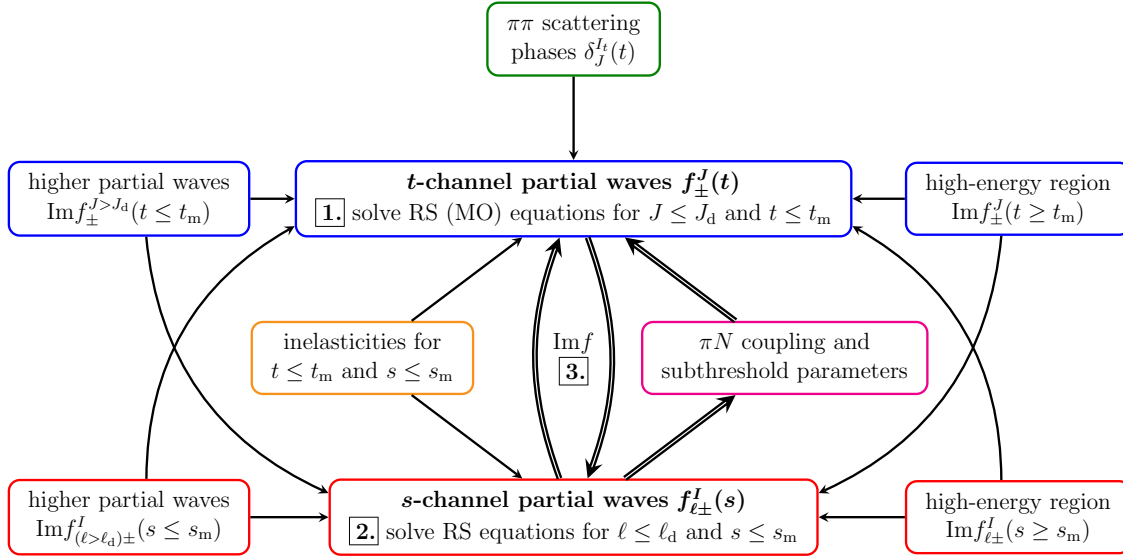


Figure 1: Flowchart of the solution strategy for the Roy–Steiner system for πN scattering. The third step consist in the self-consistent iteration (denoted by thick arrows) of the preceding steps until convergence.

where the inhomogeneities $\Delta(t)$ subsume the nucleon pole terms, all s -channel integrals, and the higher t -channel partial waves. For $t_\pi \leq t \leq t_m$, solving for $|f(t)|$ only according to Watson’s theorem requires $\delta(t)$ for $t_\pi \leq t \leq t_m$ and $\text{Im } f(t)$ for $t \geq t_m$. Introducing $n \geq 1$ subtractions does not change the general structure of the RS/MO system, e.g. the P -waves are given by

$$\Gamma^1(t) = \Delta_\Gamma^1(t) \Big|^{n\text{-sub}} + \frac{t^{n-1}(t-t_N)}{\pi} \int_{t_\pi}^{\infty} \frac{dt' \text{Im } \Gamma^1(t')}{t'^{n-1}(t'-t_N)(t'-t)}, \quad f_-^1(t) = \Delta_-^1(t) \Big|^{n\text{-sub}} + \frac{t^n}{\pi} \int_{t_\pi}^{\infty} \frac{dt' \text{Im } f_-^1(t')}{t'^n(t'-t)},$$

demonstrating that Γ^J and hence f_+^J is effectively subtracted by one power less than f_-^J , which motivates the additional (partial) third subtraction in A^\pm , cf. (1.2), that affects solely the f_+^J .

The solution strategy for the full RS system in the low-energy (or even subthreshold/pseudo-physical) regions, where only the lowest partial waves are relevant and inelastic contributions may be (approximately) neglected, is shown in Fig. 1; see [7] for more details.

2. The t -channel Muskhelishvili–Omnès problem: P -wave solutions

As the first step in the numerical solution of the full RS system, we check the consistency of our t -channel MO solutions with the results of the KH80 analysis [17], which are still used nowadays although no thorough error estimates are given (and despite the availability of more modern experimental data). Here, we present results for the P -waves in the (elastic) single-channel approximation of the MO problem, which is well justified for the P - and higher partial waves, whereas the S -wave requires a two-channel description including $\bar{K}K$ intermediate states as described in [7]. To produce the results (that will also serve as input for the solution of the s -channel RS subsystem, cf. Fig. 1) partly shown in Fig. 2, we have used as input $\pi\pi$ phase shifts from [18], s -channel partial waves ($l \leq 4$) from SAID [19] for $W \leq 2.5 \text{ GeV}$, and above the Regge model of [20]. To facilitate

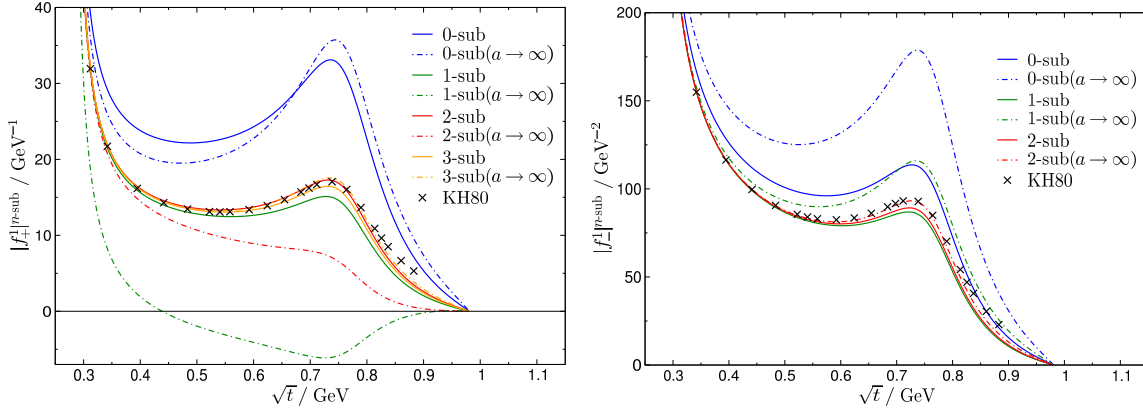


Figure 2: n -subtracted MO solutions for the P -wave moduli.

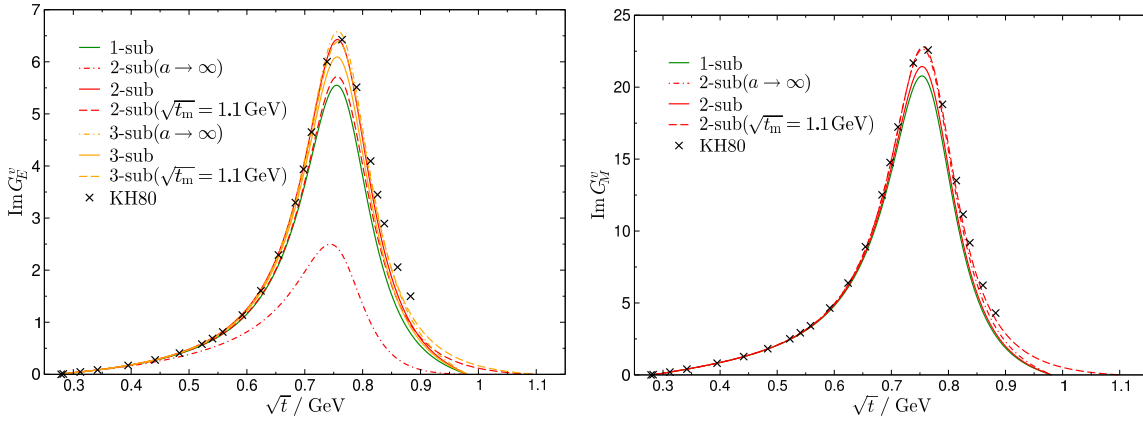


Figure 3: Two-pion-continuum contribution to $\text{Im } G_E^v$ and $\text{Im } G_M^v$.

comparison with the results of KH80, we use the respective subthreshold parameter values and a πN coupling of $g^2/(4\pi) = 14.28$ [6, 17] (as starting point, the final values will result from the iteration procedure, cf. Fig. 1).² Moreover, KH80 uses different types of dispersion relations, in particular so-called fixed- t ones, which can be emulated (up to the t -channel contributions that are not present at all in the fixed- t case) by taking the “fixed- t limit” $|a| \rightarrow \infty$. As argued in [7], all t -channel input above $\sqrt{t_m} = 0.98 \text{ GeV}$ is set to zero, which forces the MO solutions to match zero at $t = t_m$. While Fig. 2 displays the results for $|a| \rightarrow \infty$, investigating the effect of using a different (i.e. higher) matching point leads to the same conclusion: with increasing number of subtractions, thus lowering the dependence on the high-energy input by introducing more subthreshold parameter contributions as subtraction polynomials, the solutions show a nice convergence pattern both in general (proving the internal consistency and numerical stability of our RS/MO framework) and in particular towards the KH80 results (being consistent with relying on KH80 values for g and the subtraction parameters). The P -waves feature prominently in the dispersive analysis of the nucleon electromagnetic form factors, see e.g. [21] and references therein, and in Fig. 3 we illustrate the effects on the spectral functions (by approximating the vector pion form factor F_π^V via a

²Modern analyses yield significant smaller values for the πN coupling, cf. e.g. $g^2/(4\pi) = 13.7 \pm 0.2$ of [13].

twice-subtracted Omnès representation, cf. [7])

$$\text{Im } G_E^V(t) = \frac{t(\sigma_t^\pi)^3}{8m} (F_\pi^V(t))^* f_+^1(t) \theta(t - t_\pi), \quad \text{Im } G_M^V(t) = \frac{t(\sigma_t^\pi)^3}{8\sqrt{2}} (F_\pi^V(t))^* f_-^1(t) \theta(t - t_\pi).$$

We are confident that a self-consistent iteration procedure between the solutions for the s - and t -channel eventually will yield a consistent and precise description (including error estimates) of the low-energy πN scattering amplitude in all kinematical channels.

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